

SINGULARITIES IN ARBITRARY CHARACTERISTIC VIA JET SCHEMES

SHIHOKO ISHII AND ANA J. REGUERA

ABSTRACT. This paper summarizes the results at the present moment about singularities with respect to the Mather-Jacobian log discrepancies over algebraically closed field of arbitrary characteristic. The basic point is the Inversion of Adjunction with respect to Mather-Jacobian discrepancies holds in arbitrary characteristic. Based on this fact we will reduce many geometric properties of the singularities into the problem on jet schemes and try to avoid discussions which are distinctive for characteristic 0.

1. INTRODUCTION

Over the base field of characteristic 0, canonical singularities and log canonical singularities play important roles in birational geometry and these are recognized “good singularities” which we admit on a minimal model. These singularities have good descriptions by jet schemes when these are locally a complete intersection.

On the other hand, [3] and [13] introduced independently singularities which have good descriptions by jet schemes in general. These are based on Mather-Jacobian discrepancies (we say MJ discrepancies for short). We define MJ-canonical (resp. MJ-log canonical) singularities based on MJ discrepancy in the similar way as canonical (resp. log canonical) singularities. These MJ-canonical and MJ-log canonical singularities have good properties: rationality (MJ-canonical), stability under small deformations (see [8]), MJ-multiplier ideals (see [7]).

Now, in turn thinking of a singularity over the field of positive characteristic, one can define canonical and log canonical singularities as well as MJ-canonical and MJ-log canonical singularities. But, canonical and log canonical singularities of positive characteristic are difficult to treat, because the following are not available at this moment:

- (1) resolution of singularities
- (2) Bertini’s second theorem (generic smoothness)
- (3) Kodaira vanishing theorem,

which are used in the discussion of singularities in characteristic 0.

For example, even an easy looking statement “a general hyperplane section of a quasi-projective variety with at worst canonical singularities has also at worst canonical singularities” is not yet proved for positive characteristic case. This statement is proved in characteristic 0 case by using resolution of the singularities and Bertini’s second theorem. If the variety is of dimension 3 over the base field of positive characteristic, then a resolution of singularities exists [2], but still this problem is not yet proved. (There is a result under certain conditions [10]).

On the other hand MJ-version of this problem has possibility to be solved. Because MJ-singularities have good description by jet scheme, which would play the alternative role of Bertini’s second theorem. As an evidence, in Corollary 4.10, we proved this statement for 3-dimensional MJ-canonical quasi projective variety, which show the possibility of MJ-discrepancy for positive characteristic discussion.

One of the aims of this paper is to summarize the results on MJ-singularities in positive characteristic case and to clarify what are proved and what are not yet proved. We think that it make sense, because there is no such paper else at the present moment and this paper must be useful for the further research on singularities of positive characteristic. The basic theorems which show that MJ-singularities have good descriptions by jet schemes are Theorem 3.14 and Theorem 3.18. These theorems are proved in the same line as in the proof of the theorems in characteristic 0 case, with carefully avoiding to use resolutions of the singularities.

The other aim is to show an evidence that jet scheme is useful to study singularities over positive characteristic base field (Corollary 4.10).

The structure of this paper is as follows: In Section 2, we give a characteristic free discussions on MJ-discrepancies. In Section 3, we give preliminaries on jet schemes and give the definition of codimension of a cylinder and a description of minimal log MJ discrepancies by jet schemes in arbitrary characteristic. Note that these were already established for characteristic 0 case. In Section 4, we show some properties of MJ-canonical and MJ-log canonical singularities in positive characteristic which are now proved and list some open problems for positive characteristic case.

Acknowledgement The authors express their hearty thanks to Masayuki Hirokado for providing them with informations on varieties over the base field of positive characteristic.

2. MATHER-JACOBIAN DISCREPANCY

Throughout this paper a variety means a reduced pure-dimensional scheme of finite type over an algebraically closed field k of arbitrary characteristic, unless otherwise stated.

Definition 2.1. Let X be a variety. We call a morphism $\varphi : Y \rightarrow X$ a *partial resolution*, if φ is proper birational and Y is normal. We sometimes call Y a partial resolution. A prime divisor over X is a prime divisor which appears on a partial resolution of X . A prime divisor E over X is called an *exceptional prime divisor*, if a partial resolution on which E appears is not isomorphic at the generic point of E .

Definition 2.2. Let E be a prime divisor over X , the Mather discrepancy $\widehat{k}_E \in \mathbb{Z}_{\geq 0}$ and Jacobian discrepancy j_E at E are defined as follows:

Let $\varphi : Y \rightarrow X$ be a partial resolution of X such that E appears on Y . Replacing Y by the normalization of the main part of the fiber product $Y \times_X \widehat{X}$, we may assume that $\varphi : Y \rightarrow X$ factors through the Nash blow up $\widehat{X} \rightarrow X$. Here, the main part means the union of irreducible components of the fiber product, each of which dominates an irreducible component of X . By the universality of the Nash blow up, the image Im of the following homomorphism is invertible:

$$\varphi^*(\wedge^d \Omega_X) \rightarrow \omega_Y,$$

where $d = \dim X$. Restricting the above homomorphism on the smooth locus Y_{sm} , we can describe

$$Im|_{Y_{sm}} = \mathcal{I}\omega_{Y_{sm}}$$

with some invertible ideal sheaf \mathcal{I} , since $\omega_{Y_{sm}}$ is invertible. As Y is normal, the generic point η of E is in Y_{sm} . Define

$$\widehat{k}_E = \text{val}_E(\mathcal{I})$$

and call it the Mather discrepancy of X at E . We note that if we express $\mathcal{I} = \mathcal{O}_Y(-\widehat{K}_{Y/X})$ with a Cartier divisor $\widehat{K}_{Y/X}$ on Y_{sm} , then \widehat{k}_E is the coefficient of $\widehat{K}_{Y/X}$ at E .

For a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, define

$$\text{ord}_E(\mathfrak{a}) = \min\{\text{val}_E(f) \mid f \in \mathfrak{a}\}.$$

Note that if $\varphi : Y \rightarrow X$ factors through the blow up of X by the ideal \mathfrak{a} , then $\text{ord}_E(\mathfrak{a})$ is the coefficient at E of the divisor Z defined by $\mathcal{O}_Y(-Z) = \mathfrak{a}\mathcal{O}_Y$. In particular, when \mathfrak{a} is the Jacobian ideal \mathcal{J}_X of X , we define

$$j_E = \text{ord}_E(\mathcal{J}_X)$$

and call it the Jacobian discrepancy of X at E .

Then, the difference $\widehat{k}_E - j_E$ is called the Mather-Jacobian discrepancy of X at E .

Definition 2.3. Let X be a variety over k , $\mathfrak{a} \subset \mathcal{O}_X$ a non-zero coherent ideal sheaf and n a non-negative real number. A real number

$$a_{MJ}(E; X, \mathfrak{a}^n) = \widehat{k}_E - j_E - n \cdot \text{ord}_E(\mathfrak{a}) + 1$$

is called the MJ-log-discrepancy of the pair (X, \mathfrak{a}^n) at E .

A pair (X, \mathfrak{a}^n) is called MJ-canonical (resp. MJ-log-canonical) at a (not necessarily closed) point $x \in X$, if

$$a_{MJ}(E; X, \mathfrak{a}^n) \geq 1 \quad (\text{resp. } \geq 0)$$

for every exceptional prime divisor E over X with the center containing x in X .

A pair (X, \mathfrak{a}^n) is called MJ-klt at a (not necessarily closed) point $x \in X$, if

$$a_{MJ}(E; X, \mathfrak{a}^n) > 0$$

for every prime divisor E over X with the center on X containing x .

We call a pair (X, \mathfrak{a}^n) MJ-canonical (resp. MJ-log-canonical, MJ-klt) if it is MJ-canonical (resp. MJ-log-canonical, MJ-klt) at every point of X .

Remark 2.4. (1) Note that the definitions of MJ-canonical and MJ-log-canonical require the condition only for “exceptional” prime divisors, while the definition of MJ-klt requires the condition for all prime divisors over X . However, about MJ-log-canonical case, if (X, \mathfrak{a}^n) is MJ-log-canonical at a point $x \in X$, then $a_{MJ}(E; X, \mathfrak{a}^n) \geq 0$ holds for every prime divisor over X with the center containing x .

(2) If X is normal and locally a complete intersection, the image of the canonical map $\wedge^d \Omega_X \rightarrow \omega_X$ is $\mathcal{J}_X \omega_X$ (see for example [6, Remark 9.6]). Then, in this case $\widehat{k}_E - j_E = k_E$ for every prime divisor E over X . Therefore, MJ-canonical and MJ-log canonical are equivalent to usual canonical and log canonical, respectively.

Definition 2.5. Let X be a variety over k . the Mather-Jacobian minimal log discrepancy (MJ-mld, for short) of the pair (X, \mathfrak{a}^n) at a proper closed subset $W \subset X$ and at a point $\eta \in X$ are defined as follows:

$$\begin{aligned} \text{mld}_{MJ}(W; X, \mathfrak{a}^n) &= \inf\{a_{MJ}(E; X, \mathfrak{a}^n) \mid E \text{ is a prime exceptional divisor} \\ &\quad \text{over } X \text{ with the center in } W\} \\ \text{mld}_{MJ}(\eta; X, \mathfrak{a}^n) &= \inf\{a_{MJ}(E; X, \mathfrak{a}^n) \mid E \text{ is a prime exceptional divisor} \\ &\quad \text{over } X \text{ with the center } \overline{\{\eta\}}\}, \end{aligned}$$

when $\dim X \geq 2$. When $\dim X = 1$ and the right hand side is ≥ 0 , then we define mld_{MJ} by the right hand side. Otherwise we define $\text{mld}_{MJ} = -\infty$.

- Remark 2.6.** (i) We strictly distinguish “center in Z ” and “center Z ”.
- (ii) For a point $x \in X$, the pair (X, \mathfrak{a}^n) is MJ-canonical (resp. MJ-log-canonical) if and only if $\text{mld}_{\text{MJ}}(\eta; X, \mathfrak{a}^n) \geq 1$ (resp. ≥ 0) for every $\eta \in X$ such that $x \in \overline{\{\eta\}}$.
- (iii) When $\dim X \geq 2$, we can prove that $\text{mld}_{\text{MJ}}(W; X, \mathfrak{a}^n) = -\infty$, if $\text{mld}_{\text{MJ}}(W; X, \mathfrak{a}^n) < 0$.
- (iv) If X is normal and of a complete intersection, then MJ-log-discrepancy coincides with usual log-discrepancy. Therefore, in this case MJ-canonical and MJ-log canonical coincide with usual canonical and log-canonical, respectively. In particular, if X is non-singular $\text{mld}_{\text{MJ}}(W; X, \mathfrak{a}^n) = \text{mld}(W; X, \mathfrak{a}^n)$, where the right hand side is the usual minimal log discrepancy.

3. THE ARC SPACE AND JET SCHEMES OF A VARIETY

Throughout this section, X is always a d -dimensional variety over an algebraically closed field k of arbitrary characteristic. In this section, we prove the Inversion of Adjunction of minimal MJ-log discrepancies by means of discussions of arc spaces and jet schemes. When the base field is of characteristic zero, this theorem was independently proved in [3] and [13] based on the idea of [6]. Here, we will give a characteristic free proof for this theorem.

Definition 3.1. For a variety X over k , let X_m ($m \in \mathbb{N}$) and X_∞ be the m -jet scheme and the arc space of X . Denote the canonical truncation morphisms by $\psi_m : X_\infty \rightarrow X_m$ and $\pi_m : X_m \rightarrow X$. In particular we denote the extremal morphism $\psi_0 = \pi_\infty : X_\infty \rightarrow X$ by π . We also denote the canonical truncation morphism $X_{m'} \rightarrow X_m$ ($m' > m$) by $\psi_{m',m}$. When we should specify the space X , we write $\psi_{m',m}^X$.

Definition 3.2. The pull back $\psi_m^{-1}(S) \subset X_\infty$ of a constructible set $S \subset X_m$ ($m \in \mathbb{Z}_{\geq 0}$) is called a cylinder.

A subset $C \subset X_\infty$ is called a thin set, if there is a closed subset $Z \subset X$ with dimension less than the dimension X such that $C \subset Z_\infty$.

A subset $C \subset \psi_m^{-1}(S)$ is called an irreducible component of the cylinder $\psi_m^{-1}(S)$ if it is a maximal irreducible closed subset of the cylinder. Here, we note that a maximal irreducible closed subset of a cylinder exists, because the set consisting irreducible closed subsets in the cylinder is an inductive set with respect to the inclusion order (existence of a maximal irreducible component follows from Zorn’s lemma which we always assume). If $\text{char } k = 0$ or $\dim X \leq 3$, then every cylinder in X_∞ has only finite number of irreducible components (the proof of this result uses resolution of singularities [2]).

Definition 3.3. For an arc $\gamma \in X_\infty$ the order of an ideal $\mathfrak{a} \subset \mathcal{O}_X$ measured by γ is defined as follows: Let $\gamma^* : \mathcal{O}_X \rightarrow k[[t]]$ be the corresponding ring homomorphism of γ . Then, we define

$$\text{ord}_\gamma(\mathfrak{a}) = \sup\{r \in \mathbb{Z}_{\geq 0} \mid \gamma^*(\mathfrak{a}) \subset (t^r)\},$$

We define subsets “contact loci” in the arc space and s -jet scheme as follows:

$$\text{Cont}^m(\mathfrak{a}) = \{\gamma \in X_\infty \mid \text{ord}_\gamma(\mathfrak{a}) = m\}$$

In the similar way, we define

$$\text{Cont}^{\geq m}(\mathfrak{a}) = \{\gamma \in X_\infty \mid \text{ord}_\gamma(\mathfrak{a}) \geq m\}$$

By the definition, we can see that

$$\text{Cont}^{\geq m}(\mathfrak{a}) = \psi_{m-1}^{-1}(Z(\mathfrak{a})_{m-1}),$$

Here $Z(\mathfrak{a})$ is the closed subscheme defined by the ideal \mathfrak{a} in X . Therefore contact loci are cylinders in X_∞ .

Definition 3.4. Let E be a prime divisor over X and $\varphi : Y \rightarrow X$ a partial resolution of X on which E appears. Let $p \in E \cap Y_{reg}$ be the generic point. We define $C_X(\text{val}_E) = \overline{\varphi_\infty((\pi_\infty^Y)^{-1}(p))}$, also denoted by N_E in the literatures on the Nash problem. Furthermore, for $q \in \mathbb{N}$, let $p_{q-1} \in (\pi_{q-1}^Y)^{-1}((E \cap Y_{reg})_{q-1})$ be the generic point and we define

$$C_X(q \cdot \text{val}_E) = \overline{\varphi_\infty(\psi_{q-1}^Y)^{-1}(p_{q-1})}$$

and call it the maximal divisorial set corresponding to the divisorial valuation $q \cdot \text{val}_E$. This definition is the same as that in [4] and [12] in case $\text{char} k = 0$.

An irreducible closed subset $C \subset X_\infty$ is called a divisorial set, if there exist $q \in \mathbb{N}$ and a prime divisor over X such that the generic point $\alpha \in C$ gives a divisorial valuation $q \cdot \text{val}_E$ by ord_α . A maximal divisorial set $C_X(q \cdot \text{val}_E)$ is the maximal set among all divisorial sets corresponding to the valuation $q \cdot \text{val}_E$.

Proposition 3.5 ([5, Lemma 4.1], [18, before lemma 3.2] or [19, 3.4], [6, Proposition 4.1]). *For $e \in \mathbb{Z}_{\geq 0}$ let*

$$X_\infty^e := \{\gamma \in X_\infty \mid \text{ord}_\gamma(\mathcal{J}_X) = e\} \quad \text{and} \quad X_{m,\infty}^e := \psi_m(X_\infty^e).$$

Then, for $m \geq e$ the canonical map $X_{m+1,\infty}^e \rightarrow X_{m,\infty}^e$ is a piecewise trivial fibration with fibers isomorphic to \mathbb{A}^d , where $d = \dim X$. Therefore, we also have that $X_{m+1,\infty}^{\leq e} \rightarrow X_{m,\infty}^{\leq e}$ is a piecewise trivial fibration with fibers isomorphic to \mathbb{A}^d , where we define

$$X_\infty^{\leq e} := \{\gamma \in X_\infty \mid \text{ord}_\gamma(\mathcal{J}_X) \leq e\} \quad \text{and} \quad X_{m,\infty}^{\leq e} := \psi_m(X_\infty^{\leq e}).$$

This proposition is stated in [5] under the condition that $\text{char} k = 0$, but in [18], [6] its proof was confirmed to work also for perfect fields in positive characteristic case.

Based on the previous result, in [19] *stable points* of the space of arcs were defined as follows: Let γ be a point of X_∞ (i.e. γ is a prime ideal of \mathcal{O}_{X_∞}) and let $Z(\gamma)$ be the set of zeros of γ in X_∞ , then γ is a stable point if there exists $m_0 \in \mathbb{N}$ and $G \in \mathcal{O}_{X_\infty} \setminus \gamma$, $G \in \mathcal{O}_{X_{m_0}}$, such that for $m \geq m_0$ the map $\psi_{m+1,m} : X_{m+1} \rightarrow X_m$ induces a trivial fibration $\overline{\psi_{m+1}(Z(\gamma))} \cap (X_{m+1})_G \rightarrow \overline{\psi_m(Z(\gamma))} \cap (X_m)_G$ with fiber \mathbb{A}^d , where $(X_m)_G$ is the open subset $X_m \setminus Z(G)$ of X_m .

The following is an easy consequence of the previous proposition when $\text{char} k = 0$, since in that case a cylinder has only finite number of irreducible components ([4, Proposition 3.6]) and therefore an irreducible component of a cylinder is regarded generically as a cylinder. We give here a proof which does not use the finiteness of the irreducible components of a cylinder so that the proof works also for positive characteristic case.

Lemma 3.6. *For an irreducible component C of a cylinder in X_∞ such that $C \not\subset \text{Sing}(X)_\infty$, its generic point γ is a stable point of X_∞ . In particular we have that there exists e such that*

$$C_m^{\leq e} := \psi_m(C) \cap X_{m,\infty}^{\leq e}$$

is a nonempty open subset of $\psi_m(C)$ and the codimension of $C_m^{\leq e}$ inside $X_{m,\infty}^{\leq e}$ stabilizes for $m \gg e$.

Proof. Let C be an irreducible component of a cylinder Γ in X_∞ such that $C \not\subset \text{Sing}(X)_\infty$. Let $\gamma \in C$ be the generic point, then by $\gamma \notin (\text{Sing} X)_\infty$ we have

$\text{ord}_\gamma(\mathcal{J}_X) = e > 0$. Then $C_m^{\leq e}$ is a nonempty open subset of $\psi_m(C)$. From this and Proposition 3.5 the result follows. \square

Conversely, if γ is a stable point of X_∞ then $Z(\gamma)$ is a cylinder and therefore, stable points of the space of arcs are precisely the generic points of the irreducible cylinders (equivalence (a) \Leftrightarrow (b) in [19, Corollary 3.12]). Moreover, the following finiteness property of the stable points holds ([18, Theorem 4.1]): If γ is a stable point of X_∞ then the maximal ideal $\gamma\widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, \gamma}}$ of the local ring $\widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, \gamma}}$ is finitely generated, therefore $\widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, \gamma}}$ is a Noetherian complete local ring. From this result, a Curve Selection Lemma ending at the irreducible cylinders follows.

Definition 3.7. For a variety X , let C be an irreducible component of a cylinder in X_∞ such that $C \not\subset (\text{Sing} X)_\infty$, then we define the codimension of C in X_∞ as follows:

$$\text{codim}(C, X_\infty) := \text{codim}(C_m^{\leq e}, X_{m, \infty}^{\leq e})$$

for $m \gg e$, where $e = \text{ord}_\gamma(\mathcal{J}_X)$ for a generic point $\gamma \in C$.

Let $\Gamma \subset X_\infty$ be a cylinder not contained in $\text{Sing}(X)_\infty$, then we define the codimension of the cylinder Γ in X_∞ as the minimal value of the codimensions of the irreducible components not contained in $\text{Sing}(X)_\infty$.

The following lemma was proved in [5, Lemma 3.4] and also in [6, Theorem 6.2, Lemma 6.3]. The former one was stated under the condition of characteristic zero, while the latter ones were stated under an arbitrary characteristic. Note that the statement of Lemma 6.2 in [6] assumed the properness of the morphism f , but actually the properness was not used in the proof.

Lemma 3.8 ([5], [6]). *Let $f : Y \rightarrow X$ be a birational morphism from a smooth variety Y and $f_m : Y_m \rightarrow X_m$ the induced morphism. Let $\gamma \in Y_\infty$ be any arc such that $\tau := \text{ord}_\gamma(\widehat{K}_{Y/X}) < \infty$. Then for $m \gg 0$, letting $\gamma_m = \psi_m^Y(\gamma)$, we have*

$$f_m^{-1}(f_m(\gamma_m)) \simeq \mathbb{A}^\tau.$$

Moreover, for every $\gamma'_m \in f_m^{-1}(f_m(\gamma_m))$ we have $\psi_{m, m-\tau}^Y(\gamma_m) = \psi_{m, m-\tau}^Y(\gamma'_m)$, where $\psi_{m, m-\tau}^Y : Y_m \rightarrow Y_{m-\tau}$ is the canonical truncation morphism.

Theorem 3.9. *Let E be a prime divisor over X and $g : \tilde{X} \rightarrow X$ a partial resolution of X such that E appears on \tilde{X} . Then $C_X(q \cdot \text{val}_E)$ is a cylinder of X_∞ of codimension*

$$\text{codim}(C_X(q \cdot \text{val}_E), X_\infty) = q \cdot (\text{ord}_E(\widehat{K}_{Y/X}) + 1).$$

Proof. Let $f : Y \rightarrow X$ be the restriction of g on an open subset $Y \subset \tilde{X}$ such that Y and $E \cap Y$ are smooth. Then, by using Lemma 3.8 we can prove the required equality in the same way as [4, Theorem 3.9]. Note that in Theorem 3.9 in [4], stated under the condition that $f : Y \rightarrow X$ is a resolution of the singularities of X , but actually we do not need that Y is a resolution, since the condition in Lemma 3.8 does not require the properness of f . \square

The following is a modified version of Lemma 3.8 for a blowing up f .

Lemma 3.10. *Let $B \subset X$ be an irreducible reduced closed subset of dimension s with the defining ideal I_B in X and let $\gamma \in X_\infty$ be an arc which is not contained in $\text{Sing}(X)_\infty \cup B_\infty$ and the center is a smooth closed point of B with $e := \text{ord}_\gamma(I_B) > 0$. Let $f : Y \rightarrow X$ be the blow-up with the center B and $\gamma' \in Y_\infty$ be the lifting of γ .*

Let $f_m : Y_{m,\infty} \rightarrow X_{m,\infty}$ be the morphism induced by f . Then for $m \gg 0$, letting $\gamma'_m = \psi_m^Y(\gamma')$, we have

$$\dim f_m^{-1}(f_m(\gamma'_m)) \geq (d - s - 1)e.$$

Moreover, for every $\gamma''_m \in f_m^{-1}(f_m(\gamma'_m))$ we have $\psi_{m,m-e}^Y(\gamma''_m) = \psi_{m,m-e}^Y(\gamma'_m)$.

Proof. Since the problem is local, we may assume that X is an affine scheme embedded in $A = \mathbb{A}_k^N$, B is defined by the ideal (x_1, \dots, x_{N-s}) in $A = \text{Spec} k[x_1, \dots, x_N]$ and the center of γ is the origin $0 = (0, \dots, 0) \in B \subset X \subset A$. Let $\gamma_m = \psi_m^X(\gamma)$, then by definition $f_m(\gamma'_m) = \gamma_m$. Then we can express the ring homomorphism $\gamma_m^* : k[x_1, \dots, x_N] \rightarrow k[t]/(t^{m+1})$ corresponding to γ_m as:

$$\begin{aligned} \gamma_m^*(x_i) &= \sum_{j=e}^m a_i^{(j)} t^j \quad (i = 1, \dots, N-s) \\ \gamma_m^*(x_i) &= \sum_{j=1}^m a_i^{(j)} t^j \quad (i = N-s+1, \dots, N), \end{aligned}$$

where $a_i^{(j)} \in k$. Here, by the assumption we may assume that $a_1^{(e)} \neq 0$, without loss of generality.

Let $f' : A' \rightarrow A$ be the blow up with the center B . Then A' is covered by $(N-s)$ affine spaces

$$U(i) = \text{Spec} k[x_i, \frac{x_1}{x_i}, \dots, \frac{x_{N-s}}{x_i}, x_{N-s+1}, \dots, x_N] \quad (i = 1, \dots, N-s).$$

By the assumption we may assume that $\gamma'_m \in U(1)_m$. We denote the restriction $f'_m|_{U(1)_m} : U(1)_m \rightarrow A'_m$ of $f'_m : A'_m \rightarrow A'_m$ by the same symbol f'_m . Let $y_i = \frac{x_i}{x_1}$ for $i = 2, \dots, N-s$, then $x_1, y_2, \dots, y_{N-s}, x_{N-s+1}, \dots, x_N$ form a coordinate system of $U(1)$. We can express $\gamma'_m \in U(1)_m$ as follows:

$$\begin{aligned} \gamma_m'^*(x_1) &= \sum_{j=e}^m a_1^{(j)} t^j \\ \gamma_m'^*(y_i) &= \sum_{j=0}^m b_i^{(j)} t^j \quad (i = 2, \dots, N-s) \\ \gamma_m'^*(x_i) &= \sum_{j=1}^m a_i^{(j)} t^j \quad (i = N-s+1, \dots, N), \end{aligned}$$

where $b_i^{(j)}$'s satisfy $\gamma_m'^*(y_i) \gamma_m'^*(x_1) = \gamma_m'^*(x_i) = \sum_{j=e}^m a_i^{(j)} t^j$ for $i = 2, \dots, N-s$.

Then, we can see that a jet $\alpha \in (f'_m)^{-1}(\gamma_m)$ is expressed by

$$\begin{aligned} \alpha^*(x_1) &= \sum_{j=e}^m a_1^{(j)} t^j \\ \alpha^*(y_i) &= \sum_{j=0}^{m-e} b_i^{(j)} t^j + \sum_{j=m-e+1}^m y_i^{(j)} t^j \quad (i = 2, \dots, N-s) \\ \alpha^*(x_i) &= \sum_{j=1}^m a_i^{(j)} t^j \quad (i = N-s+1, \dots, N), \end{aligned}$$

where $y_i^{(j)}$ ($i = 2, \dots, N-s, j = m-e+1, \dots, m$) can be an arbitrary element in k . Therefore we have that every $\alpha \in (f'_m)^{-1}(\gamma_m)$ is mapped to the same element $\psi_{m,m-e}^{A'}(\gamma'_m)$ by the truncation morphism $\psi_{m,m-e}^{A'}$, in particular, which yields the second statement of the lemma.

Let $\gamma'_{m-e} = \psi_{m,m-e}^{A'}(\gamma'_m)$, then any element $\beta \in (\psi_{m,m-e}^{A'})^{-1}(\gamma'_{m-e})$ is expressed as

$$\begin{aligned}\beta^*(x_1) &= \sum_{j=e}^{m-e} a_1^{(j)} t^j + \sum_{j=m-e+1}^m x_1^{(j)} t^j \\ \beta^*(y_i) &= \sum_{j=0}^{m-e} b_i^{(j)} t^j + \sum_{j=m-e+1}^m y_i^{(j)} t^j \quad (i = 2, \dots, N-s) \\ \beta^*(x_i) &= \sum_{j=1}^{m-e} a_i^{(j)} t^j + \sum_{j=m-e+1}^m x_i^{(j)} t^j \quad (i = N-s+1, \dots, N),\end{aligned}$$

where $y_i^{(j)}$ and $x_i^{(j)}$ are arbitrary element of k . Here, comparing the expressions of α^* and β^* above, we obtain that

$$(f'_m)^{-1}(\gamma_m) = (f'_m)^{-1}(\gamma_m) \cap (\psi_{m,m-e}^{A'})^{-1}(\gamma'_{m-e}) \subset (\psi_{m,m-e}^{A'})^{-1}(\gamma'_{m-e})$$

is defined by $(s+1)e$ equations:

$$\begin{aligned}x_1^{(j)} &= a_1^{(j)}, (j = m-e+1, \dots, m) \\ x_i^{(j)} &= a_i^{(j)}, (i = N-s+1, \dots, N, \quad j = m-e+1, \dots, m)\end{aligned}$$

in $(\psi_{m,m-e}^{A'})^{-1}(\gamma'_{m-e})$.

Now remember that Y is the strict transform of X in A' , \mathcal{J}_Y be the Jacobian ideal of Y and let $r := \text{ord}_{\gamma'}(\mathcal{J}_Y)$. Take m sufficiently big with respect to r and e . Let $\rho_{m,m-1} : Y_{m,\infty}^{\leq r} \rightarrow Y_{m-1,\infty}^{\leq r}$ be the restriction of the truncation morphism $\psi_{m,m-1}^Y : Y_m \rightarrow Y_{m-1}$. Then, $\rho_{m,m-1}$ is a piecewise trivial morphism with the fiber \mathbb{A}^d . Therefore $\rho_{m,m-e} : Y_{m,\infty}^{\leq r} \rightarrow Y_{m-e,\infty}^{\leq r}$ has exactly de -dimensional fibers. Here we note that $\rho_{m,m-e}^{-1}(\gamma'_{m-e}) \subset (\psi_{m,m-e}^{A'})^{-1}(\gamma'_{m-e})$. By this and the above observation, we obtain that

$$(f'_m)^{-1}(\gamma_m) \cap \rho_{m,m-e}^{-1}(\gamma'_{m-e}) \subset \rho_{m,m-e}^{-1}(\gamma'_{m-e})$$

is defined by at most $(s+1)e$ equations in de -dimensional variety $\rho_{m,m-e}^{-1}(\gamma'_{m-e})$. This implies that $f_m : Y_m \rightarrow X_m$ has a fiber $f_m^{-1}(\gamma_m)$ of dimension $\geq (d-s-1)e$. \square

Remark 3.11. In the previous lemma, the center of the lifting $\gamma' \in Y_\infty$ may be a singular point of X , while the center of $\gamma \in Y_\infty$ is non-singular in Lemma 3.8. So the situation of Lemma 3.10 is different from that of Lemma 3.8.

As a corollary we obtain the following. But the result has been proved in [19, Proposition 3.7 (vii)] using the concept of stable points in the space of arcs [19, Definition 3.6] (see Lemma 3.6).

This corollary was also proved in [4, Proposition 2.12] for characteristic zero case by using resolutions of singularities.

Corollary 3.12. *Every irreducible component C of a cylinder such that $C \not\subset \text{Sing}(X)_\infty$ gives a divisorial valuation. In particular the closure of an irreducible component C of finite intersection of contact loci $\text{Cont}^{m_i}(\mathfrak{a}_i)$ ($i = 1, \dots, s$) is a maximal divisorial set.*

Proof. Let α be the generic point of C . First we prove that there exists a prime divisor E over X such that the lifting $\tilde{\alpha}$ of α onto Y on which E appears has the center E . Let $B \subset X$ be the center of α . If $\dim B = d-1$, then we have the required conclusion. Therefore we assume that $\dim B = s < d-1$. By replacing X by a small affine neighborhood, we may assume that X is embedded in the affine

space $A = \mathbb{A}_k^N$. Let $f^{(1)} : A^{(1)} \rightarrow A$ be the blow up with the center B . Then, $f^{(1)}$ is isomorphic outside B and the image of α is not contained in B , therefore α is lifted onto the proper transform $Y^{(1)}$ of X on $A^{(1)}$ by the properness criteria applied to $f^{(1)}$. Then the lifting $\alpha^{(1)}$ of α on $Y^{(1)}$ gives an irreducible component $C^{(1)}$ of a cylinder on $Y^{(1)}$. Indeed, if C is an irreducible component of the cylinder $\Gamma = (\psi_m^X)^{-1}(S)$ for a constructible subset $S \subset X_m$, then $C^{(1)}$ is an irreducible component of the cylinder $(\psi_m^{Y^{(1)}})^{-1}((f_m^{(1)})^{-1}(S))$.

Now taking m sufficiently big, and e as $\text{ord}_\gamma I_B$ for general $\gamma \in C$ with the center at a closed point in B , we obtain that the morphism $f^{(1)} : C_m^{(1)} \rightarrow C_m$ has relative dimension $\geq (d - s - 1)e \geq 1$. Therefore $\dim C_m^{(1)} > \dim C_m$ which yields $\text{codim}(C^{(1)}, Y_\infty^{(1)}) < \text{codim}(C, X_\infty)$. If the center of the generic point of $C^{(1)}$ is not of dimension $d - 1$, we blow up at the center and obtain the irreducible component whose codimension is less than the previous one. In this way we obtain the successive blow ups:

$$A^{(n)} \rightarrow \dots \rightarrow A^{(2)} \rightarrow A^{(1)} \rightarrow A$$

with the sequence of irreducible components

$$C^{(n)}, \dots, C^{(2)}, C^{(1)}, C$$

of cylinders in $Y^{(n)}, \dots, Y^{(2)}, Y^{(1)}, X$ such that

$$\text{codim} C^{(n)} < \dots < \text{codim} C^{(2)} < \text{codim} C^{(1)} < \text{codim} C.$$

Because the codimension is finite, this procedure should terminate, *i.e.*, there exists a number n such that the dimension of the center of the generic point of $C^{(n)}$ has dimension $d - 1$. This is the conclusion to our first claim.

Then we prove that the lifting $\alpha^{(n)}$ of α gives a divisorial valuation $q \cdot \text{val}_E$. Let E be the prime divisor such that $\alpha^{(n)}$ has the center there. The arc $\alpha^{(n)}$ gives a ring homomorphism of local rings $\alpha^{(n)*} : \mathcal{O}_{Y^{(n)}, \alpha^{(n)}(0)} \rightarrow K[[t]]$, where K is the residue field of $\alpha^{(n)}(0) \in Y^{(n)}$, where $\alpha^{(n)}(0)$ is the center of $\alpha^{(n)}$ on $Y^{(n)}$, actually it is the generic point of E . Then we can see that $\alpha^{(n)*}$ factors through

$$\beta : \widehat{\mathcal{O}}_{Y^{(n)}, \alpha^{(n)}(0)} = k(E)[[\tau]] \rightarrow K[[t]],$$

where $k(E)$ is the rational function field of E and τ is the generator of the maximal ideal of the local ring $\mathcal{O}_{Y^{(n)}, \alpha^{(n)}(0)}$. If we denote $\beta(\tau) = t^q$, then it implies that $\text{ord}_{\alpha^{(n)}} = q \cdot \text{val}_E$.

Finally we assume that C is an irreducible component of the intersection of $\text{Cont}^{m_i}(\mathfrak{a}_i)$'s. The generic point α of C gives a divisorial valuation $q \cdot \text{val}_E$ for a prime divisor over X by the above discussions. By definition of $\text{Cont}^{m_i}(\mathfrak{a}_i)$, it contains also the generic point of the maximal divisorial set $C(q \cdot \text{val}_E)$ for every i . This complete the proof of the last assertion of the corollary. \square

For the interpretation of MJ-minimal log discrepancies with center at a (not necessarily closed) point in terms of jet schemes, we need the following definition:

Definition 3.13. Let X be a variety and $\eta \in X$ a (not necessarily closed) point. For a cylinder $S \subset X_\infty$ we define the codimension of $S \cap \pi^{-1}(\eta)$ as follows:

$$\text{codim}(S \cap \pi^{-1}(\eta), X_\infty) := \inf \left\{ \text{codim} C \mid \begin{array}{l} C \text{ is an irreducible component of } S \cap \pi^{-1}(\overline{\{\eta\}}) \\ \text{dominating } \overline{\{\eta\}} \text{ and not contained in } (\text{Sing} X)_\infty \end{array} \right\}.$$

By Corollary 3.12, we obtain the following interpretation of the Mather-Jacobian minimal log discrepancy by the arc space. This is proved [13] for characteristic zero case.

Theorem 3.14. *Let (X, \mathfrak{a}) be a pair consisting of an arbitrary variety X and a non-zero coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. Let W be a proper closed subset of X and I_W be the (reduced) ideal of W . Then,*

$$\begin{aligned}
& \text{mld}_{\text{MJ}}(W; X, \mathfrak{a}^n) \\
(1) \quad &= \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2 \} \\
& \text{mld}_{\text{MJ}}(W; X, \mathfrak{a}^n) \\
(2) \quad &= \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{\geq m_1}(\mathfrak{a}) \cap \text{Cont}^{\geq m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2 \}. \\
& \text{For a (not necessarily closed) point } \eta \in X \text{ we also have} \\
& \text{mld}_{\text{MJ}}(\eta; X, \mathfrak{a}^n) \\
(3) \quad &= \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \pi^{-1}(\eta)) - m_1 n - m_2 \}. \\
& \text{mld}_{\text{MJ}}(\eta; X, \mathfrak{a}^n) \\
(4) \quad &= \inf_{m_i \in \mathbb{N}} \{ \text{codim}(\text{Cont}^{\geq m_1}(\mathfrak{a}) \cap \text{Cont}^{\geq m_2}(\mathcal{J}_X) \cap \pi^{-1}(\eta)) - m_1 n - m_2 \}.
\end{aligned}$$

Proof. For $\text{char} k = 0$, we proved these in [13, Proposition 3.8 and Remark 3.8], by using [4, Theorem 3.9]. The proof for positive characteristic case follows just in the same way by using Theorem 3.9 and Corollary 3.12. However, for the reader's convenience, we write down the proof here. For the proof of \geq at (1), let E be any prime divisor over X with the center in W . Let $m_1 = \text{ord}_E(\mathfrak{a})$, $m_2 = \text{ord}_E(\mathcal{J}_X)$ and $v = \text{val}_E$. Then, there is a non-empty open subset C of the maximal divisorial set $C_X(v)$ such that $C \subset \text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)$. Hence,

$$\begin{aligned}
& \widehat{k}_E - \text{ord}_E(\mathcal{J}_X) - n \cdot \text{ord}_E(\mathfrak{a}) + 1 = \text{codim}(C_X(v)) - m_1 n - m_2 \\
& \geq \text{codim}(\text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2,
\end{aligned}$$

which yields the required inequality unless $\dim X = 1$ and $\text{mld}_{\text{MJ}}(X, \mathfrak{a}) = -\infty$.

When $\dim X = 1$ and $\text{mld}_{\text{MJ}}(X, \mathfrak{a}) = -\infty$, there is a prime divisor E over X with the center in W such that $\widehat{k}_E - \text{ord}_E(\mathcal{J}_X) - n \cdot \text{ord}_E(\mathfrak{a}) + 1 < 0$. Let $m_1 = \text{ord}_E(\mathfrak{a})$ and $m_2 = \text{ord}_E(\mathcal{J}_X)$, then $\text{codim} C_X(\text{val}_E) - m_1 n - m_2 < 0$. Here, for every $q \in \mathbb{N}$, by Proposition 3.9,

$$\text{codim} C_X(q \cdot \text{val}_E) - q m_1 n - q m_2 = q(\text{codim} C_X(\text{val}_E) - m_1 n - m_2) < 0.$$

As a non-empty open subset of $C_X(q \cdot \text{val}_E)$ is contained in $\text{Cont}^{q m_1}(\mathfrak{a}) \cap \text{Cont}^{q m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)$, we have

$$\begin{aligned}
& \text{codim}(\text{Cont}^{q m_1}(\mathfrak{a}) \cap \text{Cont}^{q m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - q m_1 n - q m_2 \\
& \leq \text{codim}(C_X(q \cdot \text{val}_E)) - q m_1 n - q m_2 \\
& = q(\text{codim}(C_X(\text{val}_E)) - m_1 n - m_2) < 0.
\end{aligned}$$

Here, if $q \rightarrow \infty$, then we have

$$\text{codim}(\text{Cont}^{q m_1}(\mathfrak{a}) \cap \text{Cont}^{q m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - q m_1 n - q m_2 \rightarrow -\infty,$$

which implies the right hand side of (1) in the theorem is $-\infty$.

For the proof of \leq at (1), we may assume that $\widehat{k}_E - \text{ord}_E(\mathcal{J}_X) - n \cdot \text{ord}_E(\mathfrak{a}) + 1 \geq 0$ for every prime divisor E over X with the center in W . Indeed if there is a prime divisor E with the center in W and $\widehat{k}_E - \text{ord}_E(\mathcal{J}_X) - n \cdot \text{ord}_E(\mathfrak{a}) + 1 < 0$, then $\text{mld}_{\text{MJ}}(W; X, \mathfrak{a}) = -\infty$ by Remark 2.6, (iii) and therefore the required inequality is trivial.

For $m_i \in \mathbb{N}$, let $C \subset \text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)$ be an irreducible component which gives the codimension of the cylinder. Then, the closure \overline{C} is

$C_X(v)$ for some divisorial valuation v by Corollary 3.12. Let $v = q \cdot \text{val}_E$, then $m_1 = v(\mathfrak{a})$, $m_2 = v(\mathcal{J}_X)$ and E is a prime divisor over X with the center in W and

$$\begin{aligned} & \text{codim}(\text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2 \\ &= \text{codim}(C_X(v)) - m_1 n - m_2 \\ &= q(\widehat{k}_E + 1) - qn \cdot \text{ord}_E(\mathfrak{a}) - q \cdot \text{ord}_E(\mathcal{J}_X) \geq \widehat{k}_E + 1 - n \cdot \text{ord}_E(\mathfrak{a}), \end{aligned}$$

which yields the required inequality.

For the proof of (2) of the theorem, let

$$\begin{aligned} a_m &= \text{codim}(\text{Cont}^{m_1}(\mathfrak{a}) \cap \text{Cont}^{m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2, \\ b_m &= \text{codim}(\text{Cont}^{\geq m_1}(\mathfrak{a}) \cap \text{Cont}^{\geq m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) - m_1 n - m_2. \end{aligned}$$

As $\text{Cont}^m(\mathfrak{a}) \subset \text{Cont}^{\geq m}(\mathfrak{a})$, we have $a_m \geq b_m$. Therefore, it follows $\inf_m \{a_m\} \geq \inf_m \{b_m\}$.

Next we prove the converse inequality. For every $m \in \mathbb{N}$, let $C_X(v)$ be the irreducible component of $\text{codim}(\text{Cont}^{\geq m_1}(\mathfrak{a}) \cap \text{Cont}^{\geq m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W))$ that gives the codimension. Then, for $m'_1 := v(\mathfrak{a}) \geq m_1$ and $m'_2 := v(\mathcal{J}_X) \geq m_2$ we have

$$\begin{aligned} & \text{codim}(\text{Cont}^{\geq m_1}(\mathfrak{a}) \cap \text{Cont}^{\geq m_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)) \\ &= \text{codim}(\text{Cont}^{m'_1}(\mathfrak{a}) \cap \text{Cont}^{m'_2}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)). \end{aligned}$$

Hence, $b_m \geq a_{m'}$, which yields $\inf_m \{b_m\} \geq \inf_m \{a_m\}$.

The equality (3) follows in the same way, indeed one has only to be careful to replace “center in W ” by “center $\overline{\{\eta\}}$ ”.

□

Remark 3.15. Our formula can be easily extended for the combination of ideals $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_r$ instead of one ideal \mathfrak{a} . I.e., we have

$$\begin{aligned} & \text{mld}_{\text{MJ}}(W; X, \mathfrak{a}_1^{e_1} \mathfrak{a}_2^{e_2} \dots \mathfrak{a}_r^{e_r}) = \\ & \inf_{m_i \in \mathbb{N}} \left\{ \text{codim} \left(\left(\bigcap_{i=1}^r \text{Cont}^{m_i}(\mathfrak{a}_i) \right) \cap \text{Cont}^{m_{r+1}}(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W) \right) - \sum_{i=1}^r m_i e_i - m_{r+1} \right\}, \end{aligned}$$

where e_i 's are positive real numbers. Here, any of $\text{Cont}^{m_i}(\mathfrak{a}_i)$'s can be replaced by $\text{Cont}^{\geq m_i}(\mathfrak{a}_i)$. For simplicity of the notation and the proofs, we keep formulating the forthcoming formulas for one ideal only. But note that the formulas in this section are also valid under this combination form.

The description of log canonical threshold on a non-singular variety by jet schemes in positive characteristic case is obtained by Zhu [20].

Here, we will show the Inversion of Adjunction Formula for MJ-minimal log discrepancies for the base field of arbitrary characteristic. The proof follows basically according to the idea in the case of characteristic zero. For the proof we prepare some lemmas. The first one was proved in [6] for arbitrary characteristic.

Lemma 3.16 ([6, Lemma 8.4]). *Let A be a non-singular variety and $M = H_1 \cap \dots \cap H_c$ a codimension c complete intersection in A . If C is an irreducible locally closed cylinder in A_∞ such that*

$$C \subset \bigcap_{i=1}^c \text{Cont}^{\geq d_i}(H_i),$$

and if there is an arc $\gamma \in C \cap M_\infty$ with $\text{ord}_\gamma(\mathcal{J}_M) = e$, then

$$\text{codim}(C \cap M_\infty, M_\infty) \leq \text{codim}(C, A_\infty) + e - \sum_{i=1}^c d_i.$$

The following lemma is proved in Lemma 8.3 in [6] for characteristic 0 case by using a resolution of singularities. In order to give a characteristic free proof, we have only to check that the proof in [6] works under the normalized blow up instead of a resolution.

Let $\Phi : \mathbb{A}^1 \times X_\infty \rightarrow X_\infty$ be the morphism giving the canonical \mathbb{A}^1 -action on X_∞ . Note that if an arc γ has the center $x \in X$, then $\Phi(0, \gamma)$ is the constant arc over x .

Lemma 3.17 ([6]). *Let C be a non-empty cylinder on X_∞ for a variety X . If $\Phi(\mathbb{A}^1 \times C) \subset C$, then $C \not\subset (\text{Sing} X)_\infty$.*

Proof. Take an arc $\gamma \in C$ and denote $C = (\psi_m^X)^{-1}(S)$ for some constructible set $S \subset X_m$. Let $x \in X$ be the center of γ , then by \mathbb{A}^1 -invariance of C , the trivial m -jet $\mathbf{0}_m^x$ at x belongs to S . Let $f : X' \rightarrow X$ be the normalized blow up at the point x and take a non-singular point x' of X' inside $f^{-1}(x)$. Let C' be the cylinder $(\psi_m^{X'})^{-1}(\mathbf{0}_m^{x'})$, then $f_\infty(C') \subset C$. As C' is not a thin set in X'_∞ , C is not thin in X_∞ either, which gives that $C \not\subset (\text{Sing} X)_\infty$. \square

The following is the Inversion of Adjunction Formula for MJ-minimal log discrepancies which was proved for characteristic zero base field in [3] and [13] independently. We give here a proof for arbitrary characteristic case. We prepared necessary lemmas for the proof in arbitrary characteristic case, that were used in the proof in characteristic zero case. So the outline of the proof can be the same as characteristic zero case (presented in [3] and [13] which are based on [6]).

Theorem 3.18 (Inversion of Adjunction). *Let X be a variety over an algebraically closed field k of an arbitrary characteristic, A a smooth variety containing X as a closed subscheme of codimension c . Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be a coherent ideal sheaf such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is non-zero. Denote the ideal of X in A by I_X . Then, for a proper closed subset W of X we have:*

$$(5) \quad \text{mld}_{\text{MJ}}(W; X, \mathfrak{a}^n) = \text{mld}(W; A, \tilde{\mathfrak{a}}^n I_X^c).$$

For a point $\eta \in X$ we have:

$$(6) \quad \text{mld}_{\text{MJ}}(\eta; X, \mathfrak{a}^n) = \text{mld}(\eta; A, \tilde{\mathfrak{a}}^n I_X^c).$$

Proof. First we prove the inequality \geq in (5). We assume contrary and will induce a contradiction. By the assumption, there exist $e, m \in \mathbb{N}$ and an irreducible component $C \subset \text{Cont}^{\geq m}(\mathfrak{a}) \cap \text{Cont}^e(\mathcal{J}_X) \cap \text{Cont}^{\geq 1}(I_W)$ such that $\text{codim} C - mn < \text{mld}(W; A, \tilde{\mathfrak{a}}^n I_X^c)$. Then for a sufficiently big $s \in \mathbb{N}$, we obtain

$$(7) \quad (s+1)d - \dim \psi_m(C) - mn = \text{codim} C - mn < \text{mld}(W; A, \tilde{\mathfrak{a}}^n I_X^c),$$

where I_W is the defining ideal of W in X . As $\psi_s(C) \subset \text{Cont}^{\geq m}(\mathfrak{a})_s \cap \text{Cont}^{\geq 1}(I_W)_s = \text{Cont}^{\geq m}(\tilde{\mathfrak{a}})_s \cap \text{Cont}^{\geq 1}(\tilde{I}_W)_s \cap X_s$, where \tilde{I}_W is the defining ideal of W in \mathcal{O}_A , we have

$$C \subset (\psi_s^A)^{-1}(\psi_s(C)) \subset \text{Cont}^{\geq m}(\tilde{\mathfrak{a}}) \cap \text{Cont}^{\geq 1}(\tilde{I}_W) \cap \text{Cont}^{\geq s+1}(I_X) =: S,$$

where $\psi_s^A : A_\infty \rightarrow A_s$ is the truncation morphism. Now we obtain

$$\begin{aligned} (d+c)(s+1) - \dim \psi_s(C) &= \text{codim}(\psi_s(C), A_s) = \text{codim}((\psi_s^A)^{-1}(\psi_s(C), A_\infty) \\ &\geq \text{codim}(S, A_\infty) \geq \text{mld}(W; A, \tilde{\mathfrak{a}}^n I_X^c) + mn + c(s+1), \end{aligned}$$

which is a contradiction to (7).

For the converse inequality in (5), we have only to show the following claim

Claim 3.19. For a prime divisor F over A with center in W there is a prime divisor E over X with center in W , and an integer $q \geq 1$, such that

$$(8) \quad q \cdot a_{MJ}(E; X, \mathfrak{a}^n) \leq a_{MJ}(F, A, \tilde{\mathfrak{a}}^n I_X^c)$$

Actually we can prove that this induces the inequality

$$\text{mld}_{MJ}(W; X, \mathfrak{a}^n) \leq \text{mld}(W; A, \tilde{\mathfrak{a}}^n I_X^c)$$

as follows: If $\text{mld}_{MJ}(W; X, \mathfrak{a}^n) = -\infty$, then the required inequality is trivial. If $\text{mld}_{MJ}(W; X, \mathfrak{a}^n) \geq 0$ then $a_{MJ}(E; X, \mathfrak{a}^n) \geq 0$ in (8), which yields $a_{MJ}(F, A, \tilde{\mathfrak{a}}^n I_X^c) \geq \frac{1}{q} a_{MJ}(F, A, \tilde{\mathfrak{a}}^n I_X^c) \geq a_{MJ}(E; X, \mathfrak{a}^n) \geq \text{mld}_{MJ}(W; X, \mathfrak{a}^n)$ for every prime divisor F over A with center in W .

In the following, we prove Claim 3.19. Consider the maximal divisorial set

$$V = C_A(\text{val}_F) \subset A_\infty,$$

then we have

$$\text{codim}(V, A_\infty) = k_F + 1.$$

The intersection $V \cap X_\infty \subset X_\infty$ is a non-empty cylinder in X_∞ and is not contained in $(\text{Sing} X)_\infty$ by Lemma 3.17. Let C be an irreducible component of $V \cap X_\infty$ that is not contained in the arc space of the singular locus of X and the generic point $\gamma \in C$ gives the minimal value $\text{ord}_\gamma(\mathcal{J}_X) = e$ among the points in $V \cap X_\infty$. We can also assume that C has the minimal codimension among the components with $\text{ord}_\gamma(\mathcal{J}_X) = e$. Then, by Corollary 3.12, there exists a prime divisor E over X with the center in W such that the generic point γ of C gives the divisorial valuation $q \cdot \text{val}_E$ for some $q \in \mathbb{N}$. Therefore, $C \subset C_X(q \cdot \text{val}_E)$ and then we have the following inequalities:

$$(9) \quad q \cdot (\widehat{k}_E + 1) = \text{codim}(C_X(q \cdot \text{val}_E), X_\infty) \leq \text{codim}(C, X_\infty)$$

Now, by an appropriate choice of c generators of I_X we can take a complete intersection scheme M of dimension d containing X such that $\text{ord}_\gamma(\mathcal{J}_M) = e$. Then, by [6, Corollary 9.2], we have

$$\mathcal{J}_M \cdot \mathcal{O}_X \subset ((I_M : I_X) + I_X)/I_X.$$

It follows that γ belongs to the open cylinder

$$V_0 := V \cap \text{Cont}^{\leq e}(\mathcal{J}_M) \cap \text{Cont}^{\leq e}(I_M : I_X).$$

Here, we note that

$$V_0 \cap X_\infty = V_0 \cap M_\infty.$$

Indeed, by the definition of V_0 an arc $\alpha \in V_0 \cap M_\infty$ has finite order along $(I_M : I_X)$ which is a defining ideal of the union X' of the components of M other than X . This means $\alpha \notin X'_\infty$.

Here, for every $\beta \in M_\infty$ we note that $\text{ord}_\beta(\mathcal{J}_X) \leq \text{ord}_\beta(\mathcal{J}_M)$, therefore by the definition of C we obtain

$$(10) \quad \text{codim}(C, X_\infty) = \text{codim}(V_0 \cap X_\infty, X_\infty) = \text{codim}(V_0 \cap M_\infty, M_\infty).$$

Apply Lemma 3.16 to V_0 and we obtain

$$\text{codim}(V_0 \cap M_\infty, M_\infty) \leq \text{codim}(V_0, A_\infty) + e - \sum_{i=1}^c d_i,$$

where d_i 's satisfy $V_0 \subset \bigcap_{i=1}^c \text{Cont}^{\geq d_i}(H_i)$ for $M = H_1 \cap \cdots \cap H_c$. Then, as V_0 is an open subset of $V = C_A(\text{val}_F)$, the term $\sum d_i$ in the above inequality satisfies $\sum d_i \geq c \cdot \text{ord}_F(I_X)$ and the equality $\text{codim}(V_0, A_\infty) = \text{codim}(V, A_\infty)$ holds. On

the other hand, $e = \text{ord}_\gamma(\mathcal{J}_X) = q \cdot \text{val}_E(\mathcal{J}_X) = q \cdot j_E$. Therefore we have the following inequality

$$(11) \quad \text{codim}(V_0 \cap M_\infty, M_\infty) \leq \text{codim}(V_0, A_\infty) + q \cdot \text{val}_E(\mathcal{J}_X) - c \cdot \text{ord}_F(I_X).$$

Now combining (9), (10) and (11), we obtain

$$q \cdot (\widehat{k}_E - j_E + 1) \leq \text{codim}(V, A_\infty) - c \cdot \text{ord}_F(I_X).$$

Note that for any proper coherent ideal sheaf $\mathfrak{b} \subset \mathcal{O}_M$ not vanishing on any component of X , we have $\text{val}_C(\mathfrak{b}|_X) \geq \text{val}_V(\mathfrak{b})$ by the inclusion $C \subset V$. In particular, this implies that

$$q \cdot \text{ord}_E(\mathfrak{b}|_X) \geq \text{ord}_F(\mathfrak{b}),$$

which yields the inequality in Claim 3.19.

The proof of the equality (6) follows in the same way. Indeed we have only to be careful to replace “with the center in W ” by “with the center $\overline{\{\eta\}}$ ” in the proof above. \square

Remark 3.20. The formula in the theorem can be easily extended for the combination of ideals $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_r$ instead of one ideal \mathfrak{a} . I.e., we have

$$\text{mld}_{\text{MJ}}(W; X, \mathfrak{a}_1^{e_1} \mathfrak{a}_2^{e_2} \cdots \mathfrak{a}_r^{e_r}) = \text{mld}(W; A, \tilde{\mathfrak{a}}_1^{e_1} \tilde{\mathfrak{a}}_2^{e_2} \cdots \tilde{\mathfrak{a}}_r^{e_r} I_X^c),$$

where e_i 's are non-negative real numbers and $\tilde{\mathfrak{a}}_1, \tilde{\mathfrak{a}}_2, \dots, \tilde{\mathfrak{a}}_r$ are coherent ideal sheaves of \mathcal{O}_A such that the extensions $\mathfrak{a}_i = \tilde{\mathfrak{a}}_i \mathcal{O}_X$'s are not zero.

Corollary 3.21. *Let $\eta \in X$ be a point of a variety X of dimension d . For each $m \in \mathbb{N}$ we denote $r_m = \dim \pi_m^{-1}(x)$ for a general closed point $x \in \overline{\{\eta\}}$. Then, we have the equality:*

$$(12) \quad \begin{aligned} \text{mld}_{\text{MJ}}(\eta; X) &:= \text{mld}_{\text{MJ}}(x; X, \mathcal{O}_X) = \inf_m \left\{ (m+1)d - \dim \overline{\pi_m^{-1}(\eta)} \right\} \\ &= \inf_m \left\{ (m+1)d - (\dim \overline{\{\eta\}} + r_m) \right\}. \end{aligned}$$

Proof. Since the statement is local, we can assume that X is a closed sub variety of codimension c of a non-singular variety A . By Theorem 3.18, we have

$$\text{mld}_{\text{MJ}}(\eta; X) = \text{mld}(\eta; A, I_X^c)$$

and by Theorem 3.14, we have

$$\text{mld}(\eta; A, I_X^c) = \inf_{m \geq 0} \left\{ \text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty) - (m+1)c \right\},$$

where, for a convenience to the later use, we shift m to $m+1$ in the right hand side. By Definition 3.13, $\text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty)$ is the minimal codimension of irreducible components $C \subset (\psi_m^A)^{-1}(X_m) \cap \pi^{-1}(\overline{\{\eta\}})$ in A_∞ that dominate $\overline{\{\eta\}}$. Therefore,

$$\text{codim}(\text{Cont}^{m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty) = \text{codim}(\overline{(\pi_m^X)^{-1}(\eta)}, A_m).$$

As $\text{codim}(\overline{(\pi_m^X)^{-1}(\eta)}, A_m) = (m+1)(d+c) - \dim \overline{(\pi_m^X)^{-1}(\eta)}$, we have the first equality in the corollary. For the second equality, we have just to know that

$$\dim \overline{\pi_m^{-1}(\eta)} = \dim \overline{\{\eta\}} + r_m.$$

\square

Corollary 3.22. *Let X be a variety of dimension d and \mathfrak{a} a coherent ideal sheaf of \mathcal{O}_X . Let $V \subset W$ be two irreducible proper closed subsets of X and η_V and η_W are the generic points of V and W , respectively. Then, we have*

(i) We have the following inequality:

$$\text{mld}_{\text{MJ}}(\eta_V; X, \mathfrak{a}) \leq \text{codim}(V, X),$$

where the equality holds if and only if η_V is a regular point and $\mathfrak{a} = \mathcal{O}_X$ around η_V .

In particular, if $x \in X$ is a closed point, then

$$\text{mld}_{\text{MJ}}(x; X, \mathfrak{a}) \leq d$$

and the equality holds if and only if X is smooth at x and $\mathfrak{a} = \mathcal{O}_X$ around x . (This statement for the usual mld is Shokurov's conjecture and is not yet proved.)

- (ii) $\text{mld}_{\text{MJ}}(\eta_V; X, \mathfrak{a}) \leq \text{mld}_{\text{MJ}}(\eta_W; X, \mathfrak{a}) + \text{codim}(V, W)$,
 where the equality holds for very general V in W . Here, V is very general means that η_V is in a subset deleted a countable number of closed subsets from W . (If k is an uncountable field, then this subset is non-empty.)
- (iii) If $\text{char } k = 0$, then the equality in (ii) holds for general V in W .

Proof. First note that if $\mathfrak{a} \neq \mathcal{O}_X$ around η_V , then

$$\text{mld}_{\text{MJ}}(\eta_V; X, \mathfrak{a}) < \text{mld}_{\text{MJ}}(\eta_V; X, \mathcal{O}_X)$$

by the definition of MJ-log discrepancy. Here, by Corollary 3.21, we have

$$(13) \quad \text{mld}_{\text{MJ}}(\eta_V; X, \mathcal{O}_X) \leq (m+1)d - r_m - \dim V$$

for every $m \geq 0$. Therefore, in particular for $m = 0$, we obtain

$$\text{mld}_{\text{MJ}}(\eta_V; X, \mathcal{O}_X) \leq \text{codim}(V, X).$$

Now assume the equality in (i): $\text{mld}_{\text{MJ}}(\eta_V; X, \mathcal{O}_X) = d - \dim V$. then, by the first comment in this proof, we have $\mathfrak{a} = \mathcal{O}_X$. Consider the inequality (13) for $m = 1$, we obtain

$$(14) \quad d - \dim V \leq (d - \dim V) + d - r_1.$$

Here, we note that r_1 is the dimension of the tangent space of X at a general closed point $x \in V$. Then, the inequality (14) gives $r_1 = d$, which means that general points of V are non-singular.

For the proof of (ii), let

$$s_V(m, n) := \text{codim}(\text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq n}(\mathfrak{a}) \cap (\psi^A)^{-1}(\eta_V), A_\infty) - mc - n.$$

$$s_W(m, n) := \text{codim}(\text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq n}(\mathfrak{a}) \cap (\psi^A)^{-1}(\eta_W), A_\infty) - mc - n.$$

Then, by Theorem 3.14 and Theorem 3.18 we have

$$\text{mld}_{\text{MJ}}(\eta_V; X, \mathfrak{a}) = \inf_{m, n} s_V(m, n) \text{ and } \text{mld}_{\text{MJ}}(\eta_W; X, \mathfrak{a}) = \inf_{m, n} s_W(m, n).$$

Then, for each m, n , by an appropriate $r = r(m, n) \in \mathbb{N}$ and a k^* -invariant closed subset $S_{m, n} = \psi_r((\text{Cont}^{\geq m}(I_X) \cap \text{Cont}^{\geq n}(\mathfrak{a})) \subset A_r$, we can express

$$\begin{aligned} s_V(m, n) - s_W(m, n) &= \dim S_{m, n} \cap \pi_r^{-1}(\eta_W) - \dim S_{m, n} \cap \pi_r^{-1}(\eta_V) \\ &= (\dim W + \delta_W) - (\dim V + \delta_V), \end{aligned}$$

where δ_V and δ_W are the dimensions of general fibers of $\pi_r|_{S_{m, n} \cap \pi_r^{-1}(V)}$ and $\pi_r|_{S_{m, n} \cap \pi_r^{-1}(W)}$, respectively. As $S_{m, n}$ is k^* -invariant, the restricted morphism $\pi_r|_{S_{m, n} \setminus \sigma(A)} : S_{m, n} \setminus \sigma(A) \rightarrow A$ factors through the projective morphism $\pi'_r : (S_{m, n} \setminus \sigma(A))/k^* \rightarrow A$. Here, $\sigma(A) \subset A_r$ is the subset consisting of the trivial r -jets on A . Therefore dimension of fibers of π'_r and also $\pi_r|_{S_{m, n}}$ are upper-semi-continuous, which implies the inequality $\delta_W \leq \delta_V$. This yields

$$s_V(m, n) - s_W(m, n) \leq \text{codim}(V, W)$$

for every m, n , which yields the inequality in (ii).

For a fixed m, n , there exists a closed subset $F_{m,n} \subset W$ such that for every point $\eta_V \in W$ which is not contained in $F_{m,n}$ satisfies $\delta_V = \delta_W$. Then, for these V we obtain

$$s_V(m, n) - s_W(m, n) = \text{codim}(V, W).$$

Therefore, if η_V is not contained in $F = \bigcup_{m,n} F_{m,n}$, then we obtain the equality

$$\text{mld}_{\text{MJ}}(\eta_V; X, \mathfrak{a}) = \text{mld}_{\text{MJ}}(\eta_W; X, \mathfrak{a}) + \text{codim}(V, W).$$

For the proof of (iii), assume that $\text{chark} = 0$. Let $f : A' \rightarrow A$ be an embedded log resolution of (X, \mathfrak{a}) with at least one exceptional divisor with the center $\overline{\{\eta_W\}}$. Then there is an exceptional prime divisor $E \subset A'$ computing the $\text{mld}_{\text{MJ}}(\eta_W; X, \mathfrak{a})$. If $\eta_V \in W$ is not in the union of the lower dimensional centers of the exceptional divisors of f , then the divisor obtained by the blow up with the center $f^{-1}(V) \cap E$ computes

$$\text{mld}_{\text{MJ}}(\eta_V; X, \mathfrak{a}) = \text{mld}_{\text{MJ}}(\eta_W; X, \mathfrak{a}) + \text{codim}(V, W).$$

□

Remark 3.23. As is seen in the proof of (iii), one can see that for a positive characteristic case if (iii) in Corollary 3.22 does not hold, then it shows a counter example of the existence of resolution of singularities.

4. MJ-CANONICAL AND MJ-LOG CANONICAL SINGULARITIES

We say that X has MJ-canonical (resp. MJ-log canonical) singularities if the pair (X, \mathcal{O}_X) has MJ-canonical (resp. MJ-log canonical) singularities. We denote $\text{mld}_{\text{MJ}}(x; X, \mathcal{O}_X)$ by $\text{mld}_{\text{MJ}}(x; X)$. In this section we will study the nature of MJ-canonical singularities and MJ-log canonical singularities.

Lemma 4.1 ([8, Proposition 3.3]). *Let $x \in X$ be a closed point. If X is MJ-canonical at x , then the embedding dimension satisfies*

$$\text{emb}(X, x) \leq 2d - 1.$$

If X is MJ-log canonical at x , then the embedding dimension satisfies

$$\text{emb}(X, x) \leq 2d.$$

Proof. This is proved in [8, Proposition 3.3] in case $\text{chark} = 0$. However, the main point of the proof is the formula in Corollary 3.21 and we have seen in the proof of the corollary that the formula holds true for arbitrary characteristic. Therefore, the same proof works for the lemma in positive characteristic case too. □

Definition 4.2. We say that a variety X has a pseudo rational singularity at $x \in X$ if

- (1) X is normal around x ;
- (2) For every partial resolution $f : Y \rightarrow X$ which means a proper birational morphism with normal Y , the equality

$$f_*\omega_Y = \omega_X$$

holds around x ;

- (3) X is Cohen-Macaulay around x .

Note that if $\text{chark} = 0$ or if $\dim X = 2$, this definition is equivalent to the following:

- (1) X is normal around x ;

(2') For every resolution $f : Y \rightarrow X$ the vanishing

$$R^i f_* \mathcal{O}_Y = 0$$

holds for $i > 0$ around x .

The singularity (X, x) satisfying (1) and (2') is called a rational singularity.

Proposition 4.3. *Let a variety X have at worst MJ-canonical singularities. Then X has normal hypersurface singularities in codimension 2.*

If $\text{char} k = 0$, then X is normal and furthermore the singularities on X are rational.

Proof. The second statement is proved in [3, Theorem 7.7] and [7, Corollary 3.7] independently. About the first statement, the problem is local, we may assume that X is a closed sub variety in the affine space $A = \mathbb{A}^N$. Then, as X is MJ-canonical, it follows that $\text{mld}_{\text{MJ}}(\eta; X) = \text{mld}(\eta, A, I_X^c) \geq 0$, where I_X is the defining ideal of X in A and $c = \text{codim}(X, A)$.

The first statement is proved by blowing up A at an irreducible component of the singular locus of X and check the discrepancy of (A, I_X^c) as in the proof of Proposition 4.6. But, here we present a proof by using jet schemes discussions.

Let $\eta \in X$ be the generic point of an irreducible closed subset of codimension one. Then, as X is MJ-canonical, it follows that $\text{mld}_{\text{MJ}}(\eta; X) \geq 1$. On the other hand, we have $\text{mld}_{\text{MJ}}(\eta; X) \leq d - \dim \overline{\{\eta\}} = 1$ by Corollary 3.22. Then, we obtain that the equality in (i) in Corollary 3.22 holds, which yields that X is regular at η . Now, $\dim \text{Sing}(X) \leq d - 2$. Let $\zeta \in X$ be the generic point of an irreducible component of $\text{Sing}(X)$ of codimension 2. Then, by inequality (13), it follows

$$1 \leq \text{mld}_{\text{MJ}}(\zeta, X) \leq (m+1)d - r_m - \dim \overline{\{\zeta\}},$$

where r_m is the dimension of a general fiber of π_m . Consider the case $m = 1$, we obtain

$$(15) \quad 1 \leq \text{mld}_{\text{MJ}}(\zeta, X) \leq 2d - r_1 - (d - 2) \leq 1.$$

Here, the last inequality is showed as follows: note that r_1 is the dimension of Zariski tangent space of X at a general point, therefore $r_1 \geq d + 1$, as the point is a singular point. Therefore, all inequalities in (15) become equalities, in particular $r_1 = d + 1$, which means that a general point is a hypersurface singularity. Hence, X is a Gorenstein variety in codimension 2 and satisfies R_1 , which yields that X is normal in codimension 2 by Serre's criteria. \square

Corollary 4.4. *A 2-dimensional singularity (X, x) is MJ-canonical if and only if it is a rational double point in arbitrary characteristic.*

Proof. In case $\text{char} k = 0$, this statement is proved in [8]. The following is a characteristic free proof. As a rational double point of dimension two is a normal hypersurface singularity, therefore the canonicity in the usual sense is equivalent to MJ-canonicity. Therefore, a rational double point of dimension two is MJ-canonical.

Conversely, if (X, x) is MJ-canonical, then by Proposition 4.3, it is a normal hypersurface singularity. Therefore it is canonical in the usual sense, which yields that it is rational double. \square

Corollary 4.5. *If an MJ-canonical variety X is locally a complete intersection, then X is normal and has pseudo rational singularities. In particular for $\text{char} k = 0$ X has rational singularities.*

Proof. As X is locally a complete intersection, it is Cohen-Macaulay. By Proposition 4.3, X satisfies R_1 , therefore by Serre's criteria, it is normal. Since X is locally a complete intersection, we have that $\hat{k}_E - j_E = k_E$ (Remark 2.4) for every prime

divisor E over X . On the other hand, when we take a partial resolution $f : Y \rightarrow X$, for every prime divisor E on Y , we obtain $\text{ord}_E(K_Y - f^*K_X) = k_E = \hat{k}_E - j_E \geq 0$ by the assumption that X is MJ-canonical. Therefore, on Y we have the inequality $K_Y \geq f^*K_X$ which yields

$$(16) \quad f_*\omega_Y \supset f_*f^*\omega_X = \omega_X.$$

Now we obtain $f_*\omega_Y = \omega_X$, since the opposite inclusion of (16) is trivial. \square

Proposition 4.6. *Let a variety X have at worst MJ-log canonical singularities. Then a general point of the singular locus of codimension one is normal crossing double.*

Proof. Since the problem is local, we may assume that X is a closed sub variety in the affine space $A = \mathbb{A}^N$. Then, as X is MJ-canonical, it follows that $\text{mld}_{\text{MJ}}(\eta; X) = \text{mld}(\eta, A, I_X^c) \geq 0$, where I_X is the defining ideal of X in A and $c = \text{codim}(X, A)$. It seems to be well known that if (A, I_X^c) is log canonical, then X is a hyper surface with at worst normal crossing double singularities in codimension 1. But for sure that it holds for arbitrary characteristic, we write down the proof.

Let $S \subset X$ be an irreducible component of codimension 1 in the singular locus and let $\eta \in S$ be the generic point. Let $f : A_1 \rightarrow A$ be the blow up of A with the center S and let E_1 be the exceptional divisor dominating S . Then, we obtain

$$a(E_1; A, I_X^c) = k_{E_1} - c \cdot \text{ord}_E(I_X) + 1 \geq 0.$$

Here, we note that $k_{E_1} = c$ and $\text{ord}_E(I_X) \geq 2$, we obtain

$$0 \leq k_{E_1} - c \cdot \text{ord}_{E_1}(I_X) + 1 \leq c - 2c + 1 = -c + 1 \leq 0.$$

Therefore all equalities should hold, in particular $a(E_1; A, I_X^c) = 0$, $c = 1$ and $\text{ord}_{E_1}(I_X) = 2$. These mean that X has hypersurface double points in codimension 1.

If a general point of S is not normal crossing double, then E_1 and the proper transform X_1 contact with the order ≥ 2 along a closed subset S_1 dominating S . Let $A_2 \rightarrow A_1$ be the blow up with the center S_1 and E_2 the exceptional divisor dominating S_1 . Then, E_2 , the proper transform E'_1 of E_1 and the proper transform X_2 of X_1 still have an intersection at a closed subset S_2 of dimension $d - 1$. Now, blow up $A_3 \rightarrow A_2$ with the center S_2 , and let E_3 be the exceptional divisor dominating S_2 . Then, the log discrepancy at E_3 is

$$a_{E_3}(\mathbb{A}^{d+1}, I_X) = k_{E_3} - \text{ord}_{E_3} I_X + 1 \leq 3 - 5 + 1 = -1,$$

a contradiction. Therefore, E_1 and X_1 has the reduced intersection over general points of S , which implies that X is normal crossing double at a general point of S . \square

Theorem 4.7. *Let k be an algebraically closed field. A pair (X, B) consisting of an arbitrary variety X over k and an effective \mathbb{R} -Cartier divisor B on X satisfies*

$$(17) \quad \dim X - 1 \leq \text{mld}_{\text{MJ}}(x; X, B)$$

if and only if either

- (i) $\dim X \geq 2$, $B = 0$ and (X, x) is a compound Du Val singularity,
- (ii) $B = 0$, and (X, x) is given by:

$$xy = 0,$$

$$z^2 + xy^2 = 0 \text{ if } \text{char } k \neq 2, \text{ or}$$

$$z^2 + xy^2 + yzg(x, y) = 0 \text{ if } \text{char } k = 2, \text{ where } \text{mult}_g \geq 1 \text{ and either } g = 0 \text{ or } g(x, 0) \neq 0.$$

- (iii) (X, x) is non-singular and $0 \leq \text{mult}_x B \leq 1$.

In the cases (i) and (ii), we have $\text{mld}_{\text{MJ}}(x; X, \mathcal{J}_X) = \dim X - 1$ and in the case (iii) we have $\text{mld}_{\text{MJ}}(x; X, B) = \text{mld}(x; X, B) = \dim X - \text{mult}_x B$ and the minimal log discrepancy is computed by the exceptional divisor of the first blowup at x .

Proof. As in [14, Theorem 4.1], it suffices to show that equality in (17) holds if and only if (i) or (ii) are satisfied. Germs of varieties (X, x) , where x is a closed point, satisfying $\text{mld}_{\text{MJ}}(x, X) = \dim X - 1$ are called in [14] top singularities.

The following results, proved in [14] if $\text{char } k = 0$, are based on equality (12), hence remain true if $\text{char } k > 0$:

(R0) [14, Lemma 3.5] Let X be a d -dimensional variety and let $X' \subset X$ be a $(d - c)$ -dimensional subvariety which is defined as the zero locus of c elements of \mathcal{O}_X . Let x be a closed point in X' . If (X', x) is a top singularity, then (X, x) is a top singularity.

(R1) [14, Lemma 3.6] If X has a top singularity at x , then X is locally at x an hypersurface of multiplicity 2.

(R2) [14, Lemma 3.20] Let (X, x) be a germ of an hypersurface in \mathbb{A}_k^{d+1} of multiplicity 2 and $\tau(X, x) = 1$ (the invariant τ is introduced in [9] and is known to be an invariant of (X, x) . Here τ is the smallest possible dimension of a linear subspace V_0 of $V = kx_1 + kx_2 + \dots + kx_{d+1}$ such that the initial term inf of f lies in the sub algebra $k[V_0]$ of $k[x_1, \dots, x_{d+1}]$). Hence the equation of (X, x) gives $x_1^2 - g(x_2, \dots, x_{d+1}) \in x_1(x_1, \dots, x_{d+1})^2$. Then, if (X, x) is a top singularity, we have $d \geq 2$ and $\text{mult } g = 3$.

(R3) [14, Proposition 3.23] In the conditions of (R2) suppose also that the initial form of g has only one factor. Hence the equation of (X, x) gives

$$x_1^2 + x_2^3 + g_3(x_3, \dots, x_{d+1})x_2 + g_4(x_3, \dots, x_{d+1}) \\ \in x_1(x_3, \dots, x_{d+1})^2 + x_1x_2(x_3, \dots, x_{d+1}) + (x_1x_2^2) + x_2^2(x_3, \dots, x_{d+1})^2$$

where $\text{mult } g_i \geq i$, for $i = 3, 4, \dots$. Then, if (X, x) is a top singularity, we have either $\text{mult } g_3 = 3$ or $4 \leq \text{mult } g_4 \leq 5$.

On the other hand, in [16] pages 256-268, a characterization of pseudo rational double points over any field k is given.

Restricting to an algebraically closed field k , the following is proved: Let:

H1) $X \subset \mathbb{A}_k^3$ be a surface of multiplicity 2. Then $\tau \leq 3$ and we have:

Case I : $\tau = 3$ if and only if $(X, 0)$ has A_1 -singularity.

Case II (II a) in [16]: $\tau = 2$ is equivalent to $(X, 0)$ having A_n -singularity ($n \geq 2$) if we assume that $(X, 0)$ is pseudo rational.

If $\tau = 1$ then the equation of X gives $z^2 - G(x, y) \in zM^2$ with $\text{mult } G \geq 3$, where M is the maximal ideal at the origin. Besides, the hypothesis X pseudo rational implies:

H2) $\text{mult } G = 3$.

Let \overline{G} be the initial form of G , then one of the following cases happens:

Case III (III c in [16]): $\tau = 1$ and \overline{G} has 3 factors. This is equivalent to X being a D_4 -singularity

Case IV : $\tau = 1$ and \overline{G} is the product of a linear factor and the square of another factor. This is equivalent to (X, x) having a D_n -singularity ($n \geq 5$) if we assume that (X, x) is pseudo rational.

Case V : $\tau = 1$ and \overline{G} has only 1 factor. Then, either (X, x) is an E_6 -singularity or it can be expressed as:

$$z^2 + y^3 + \rho x^3 y + \sigma x^5 \in (zxy, zy^2, x^2 y^2, x^3 z).$$

Again (X, x) pseudo rational implies:

H3) either ρ is a unit or σ is a unit.

Finally we have:

Case V c : ρ being a unit is equivalent to (X, x) being an E_7 -singularity.

Case V d : ρ not a unit and σ a unit, is equivalent to (X, x) being an E_8 -singularity.

Note that rational double points are top singularities (apply the proof for char $k = 0$ given in [14], example 3.12). From this, applying (R0), it follows that (i) in the theorem implies equality in (16). The fact that (ii) implies equality in (16) can also be checked. In fact, for the last case in (ii), note that the surface given by $z^2 + xy^2 + yzg(x, y) = 0$ (char $k = 2$), where $\text{mult}_g \geq 1$ and either $g = 0$ or $g(x, 0) = 0$, can be desingularized by the blowing up of (y, z) and, if E is the exceptional curve appearing then $\widehat{k}_E = 1$ and $\text{ord}_E(\mathcal{J}_X) = 1$.

Finally, let us prove that if (X, x) is a top singularity then either (i) or (ii) hold. In fact, let (X', x) be the surface obtained by a general cut of (X, x) with $d - 2$ hyperplanes. Since (R1) holds for (X, x) , we have that (H1) holds for (X', x) . Analogously, since (R2) holds for (X, x) , it follows that for (X', x) we have that if $\tau = 1$ then (H2) holds. Finally, (R3) for (X, x) implies that, if (X', x) satisfies the hypothesis of Case V then either (X', x) is an E_6 -singularity or (H3) holds.

We conclude thus that, if we assume that (X', x) is pseudo rational in case II and case IV, then (X, x) being a top singularity would imply that (X', x) is a rational double point, hence (X, x) is a compound Du Val singularity.

Finally, for cases II and IV without the hypothesis of pseudo rational, we obtain:

1. If (X', x) is in case II and not pseudo rational then it can be expressed as $xy = 0$.
2. If (X', x) is in case IV and not pseudo rational then it can be expressed as:
 $z^2 + xy^2 = 0$ if $\text{char } k \neq 2$, or
 $z^2 + xy^2 + yzg(x, y) = 0$ if $\text{char } k = 2$, where $\text{mult}_g \geq 1$ and either $g = 0$ or $g(x, 0) \neq 0$.

Thus, we conclude the result. \square

Remark 4.8. The proof of the theorem shows that $s_m \geq d - 1$ ($m \leq 5$) yields that $\text{mld}_{\text{MJ}}(x; X) \geq d - 1$.

Proposition 4.9. *Let X be a variety over an algebraically closed field k . Assume that X has at worst MJ-canonical singularities. Then, X has at worst cDV singularities in codimension 2,*

Proof. In Proposition 4.3 it is proved that X has normal in codimension 2. Let $\eta \in X$ be the generic point of an irreducible component of the singular locus of codimension 2. As X is MJ-canonical, it follows $\text{mld}_{\text{MJ}}(\eta; X) \geq 1$, therefore by the Corollary 3.21

$$1 \leq \text{mld}_{\text{MJ}}(\eta; X) = \inf_m \left\{ (m+1)d - (\dim \overline{\{\eta\}} + r_m) \right\},$$

where $r_m = \dim \pi_m^{-1}(x)$ ($m \in \mathbb{N}$) for a general closed point $x \in \overline{\{\eta\}}$. Considering the cases for $m = 1, \dots, 5$, we obtain that

$$s_m(x) := (m+1)d - r_m \geq d - 1$$

for a general point $x \in \overline{\{\eta\}}$. Therefore, by Remark 4.8, we obtain that $\text{mld}_{\text{MJ}}(x; X) = d - 1$ at a general point $x \in \overline{\{\eta\}}$. \square

Corollary 4.10. *Let X be an MJ-canonical quasi-projective variety of dimension 3 over an algebraically closed field k . Then, a general hyperplane section H of X has at worst Du Val singularities.*

Proof. Let X be a locally closed sub variety of \mathbb{P}^N . As the linear system of hyperplane in \mathbb{P}^N is very ample, by old Bertini's theorem, a general hyperplane section H is non-singular away from the singular locus of X . Therefore, if $\dim \text{Sing} X = 0$, then H is non-singular. So we assume that $\dim \text{Sing} X = 1$. By Proposition 4.9 a general hyperplane section intersects the singular locus at finite number of cDV points. We have only to show that the intersection is general at each point $x \in H \cap \text{Sing} X$. Let $|\mathcal{O}_{\mathbb{P}^N}(1)|$ be the complete linear system of hyperplanes on \mathbb{P}^N . For a point $x \in \text{Sing} X$, we define a subset $D_x \subset |\mathcal{O}_{\mathbb{P}^N}(1)|$ as

$$D_x = \{\mathcal{H} \in |\mathcal{O}_{\mathbb{P}^N}(1)| \mid X \subset \mathcal{H} \text{ or } (\mathcal{H} \cap X, x) \text{ not rational double}\}.$$

As $\mathcal{O}_{\mathbb{P}^N}(1)$ is very ample, the canonical k -linear map

$$\varphi_x : \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \rightarrow \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X / \mathfrak{m}_x^2 \simeq \mathcal{O}_X / \mathfrak{m}_x^2$$

is surjective. Let $\tilde{D}_x \subset \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ be the subset corresponding to D_x for a cDV point $x \in \text{Sing} X$, then \tilde{D}_x is the pull back by φ_x of the proper closed subset of $k^4 = \mathfrak{m}_x / \mathfrak{m}_x^2 \subset \mathcal{O}_X / \mathfrak{m}_x^2$. Then,

$$\dim \tilde{D}_x \leq N + 1 - 5 + 3 = N - 1,$$

therefore

$$\dim D_x \leq N - 2.$$

On the other hand, if $x \in \text{Sing} X$ is not a cDV point, then

$$\dim D_x \leq N - 1.$$

Let $D \subset \text{Sing} X \times |\mathcal{O}_{\mathbb{P}^N}(1)|$ be the set $\{\langle x, \mathcal{H} \rangle \mid \mathcal{H} \in D_x\}$. Then, as $\text{Sing} X$ is of 1-dimensional and non cDV singularities are isolated, therefore we have

$$\dim D \leq N - 1.$$

Hence, the image $p(D) \subset |\mathcal{O}_{\mathbb{P}^N}(1)|$ of D by the projection

$$p : \text{Sing} X \times |\mathcal{O}_{\mathbb{P}^N}(1)| \rightarrow |\mathcal{O}_{\mathbb{P}^N}(1)|$$

has dimension $N - 1 \leq N = \dim |\mathcal{O}_{\mathbb{P}^N}(1)|$, which yields that general elements of $|\mathcal{O}_{\mathbb{P}^N}(1)|$ is not in $p(D)$. This completes the proof. \square

The usual canonical version of this statement is proved by [10] under some conditions.

Here, we list open problems for positive characteristic case, which are all proved for characteristic zero (see [7] for (1) and see [8] for (2),(3),(4)):

Open problems for positive characteristic case:

- (1) Is MJ-canonical singularity normal? Cohen-Macaulay?
- (2) Is the map $X \rightarrow \mathbb{Z}, x \mapsto \text{mld}_{\text{MJ}}(x, X)$ lower semi-continuous?
- (3) Is MJ-canonical (resp. MJ-log canonical) singularity open condition?
- (4) Is a small deformation of MJ-canonical singularity again MJ-canonical?

Here, we note that if there exist resolutions of singularities, then we would have the affirmative answer to (2),(3) and (4). On the other hand, without resolutions we can prove these if the following natural conjecture holds:

Conjecture 4.11. *Let $0 \leq \delta \leq d$, there is a number $N_{\delta,d}$ depending only on δ and d such that if*

$$s_m(x) := (m+1)d - \dim \pi_m^{-1}(x) \geq \delta, \text{ for all } m \leq N_{\delta,d}$$

then $\text{mld}_{\text{MJ}}(x, X) \geq \delta$.

We observe that this holds true for $\delta = d - 1$. Indeed, we can take $N_{d-1,d} = 5$ as is seen in the proof of Remark 4.8.

This conjecture is equivalent to the following conjecture:

Conjecture 4.12. *There exists a number N_d depending only on d such that*

$$\min\{s_m(x) \mid m \leq N_d\} = \text{mld}_{\text{MJ}}(x; X).$$

Remark 4.13. We stated the results in this paper under the condition that k is algebraically closed. But we can weaken this condition to that k is perfect in all results except for Theorem 4.7 and Corollary 4.10.

REFERENCES

1. M. Artin, *Coverings of the rational double points in characteristic p* , Complex Analytic and Algebraic Geometry, Iwanami and Cambridge UP, (1977) 12–22.
2. V. Cossart, O. Piltant, *Resolution of singularities of threefolds in positive characteristic. II Artin-Schreier and purely inseparable coverings*, J. Algebra **321** (2009), no.1, 1836–1976.
3. T. De Fernex, R. Docampo *Jacobian discrepancies and rational singularities*, J. Eur. Math. Soc. **16** (2014), 165–199.
4. T. De Fernex, L. Ein and S. Ishii, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. **44** no. 2, 425–448 (2008).
5. J. Denef and F. Loeser, *Germes of arcs on singular algebraic varieties and motivic integration*. Invent. Math. **135** (1999), 201–232.
6. L. Ein and M. Mustařă, *Jet schemes and singularities*, Proc. Symp. Pure Math. **80.2**, 505–546 (2009).
7. L. Ein, S. Ishii and M. Mustařă, *Multiplier ideals via Mather discrepancy*, to appear in Adv. Studies in Pure Math..
8. L. Ein and S. Ishii, *Singularities with respect to Mather-Jacobian discrepancies*, preprint (2013) ArXiv:1310.6882. to appear in Publ. MSRI.
9. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I and II*, Annals of Maths. **79** ns.1 and 2, 109–326 (1964).
10. M. Hirokado, M. Ito and N. Saito, *Three dimensional canonical singularities in codimension two in positive characteristic*, J. Algebra **373** (2013) 207–222.
11. S. Ishii, *Jet schemes, arc spaces and the Nash problem*, C.R.Math. Rep. Acad. Canada, **29** (2007) 1–21.
12. S. Ishii, *Maximal divisorial sets in arc spaces*, Adv. St. Pure Math. **50**, (2008) 237–249.
13. S. Ishii, *Mather discrepancy and the arc spaces*, Ann. de l’Institut Fourier, **63**, (2013) 89–111.
14. S. Ishii and Ana Reguera, *Singularities with the highest Mather minimal log discrepancy*, Math. Zeitschrift., **275**, Issue 3-4, pp (2013)1255–1274.
15. E. R. Kolchin, *Differential algebra and algebraic groups*, Pure and Applied Mathematics, Vol. **54**, Academic Press, New York-London, 1973.
16. J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Publ. Math. I.H.E.S. **36**, 195–279 (1969).
17. J. Nicaise and J. Sebag, *Le th  or  me d’irr  ductibilit   de Kolchin*, C. R. Math. Acad. Sci. Paris **341** (2005), 103–106.
18. A.J. Reguera, *A curve selection lemma in spaces of arcs and the image of the Nash map*, Compositio Math. **142**, 119–130, (2006).
19. A.J. Reguera, *Towards the singular locus of the space of arcs*, Amer. J. Math. **131**, 2, 313–350, (2009).
20. Z. Zhu, *Log canonical thresholds in positive characteristic*, preprint, arXiv:1308.5445.

Shihoko Ishii,
Graduate School of Mathematical Science, University of Tokyo,
3-8-1 Komaba, Meguro, 153-8914 Tokyo, Japan.
E-mail: shihoko@ms.u-tokyo.ac.jp

Ana J. Reguera,
Dep. de Álgebra, Geometría y Topología, Universidad de Valladolid,
Paseo Belén 7, 47011 Valladolid, Spain.
E-mail: areguera@agt.uva.es